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RELATIONSHIP M(κ) ON THE BASIS OF RESEARCH ON BEAMS SUBJECTED TO CHANGING LOADS

Abstract

While calculating reinforced concrete constructions the relationship momentum – curvature M(κ) has significant importance. In exploitation conditions reinforced concrete construction is subjected to increasing loading and unloading. In bent moments, performing even a few cycles of static loading and unloading causes increase in temporary deflection. Building a unified model of relationship M(κ) describing the whole range from the elastic state to the full plasticisation, allowing for variable reversible loads (hysteresis loop) requires further investigations. Degradation of cross section stiffness is the result of damage process and plasticisation. Unified description of processes taking place in reinforced concrete beam cross section can be obtained with the use of elastic work probability, plastic work probability and damage relating cross section stiffness.

Keywords: concrete beams, changing loads

1. Introduction

The relationship moment – curvature $\boldsymbol{M}(\kappa)$ has special meaning for calculating reinforced concrete constructions. It determines distribution of internal force in static non-determinable states and measures of element deflections. The relationships $\boldsymbol{M}(\kappa)$ presented in papers [1-4] allows to calculate reinforced concrete loaded with monotonically increasing instantaneous or long-lasting load.

Theoretical recognition of the relationship $\boldsymbol{M}(\kappa)$in the process of loading by basic methods can be found in ([5], [6]). Obtained numerically theoretical relationships are strongly determined by the assumption of linear-elastic behaviour of unloaded concrete and steel with the unloading modules equal to the initial values in the process of loading. Applying the simplified material relationships for concrete cross section behaviour neglects specific features of reinforced concrete cross section behaviour in the process of unloading. In the reinforced concrete cross section appear mechanisms related to micro and macro cracks [7], plasticisation and self-stress [8].

In elements subjected to bending, performing even a few cycles of static loading and unloading causes increase in temporary deflection. Building a unified model of $\boldsymbol{M}(\kappa)$ describing the whole range from the elastic state to the full plasticisation allowing for variable reversible loads (hysteresis loop) required relevant investigation.

The purpose of this paper is to present a proposal of constitutive relationship that is valid not only in the process of increasing loads but unloads as well, showing its real characteristics. Extending the relationship $\boldsymbol{M}(\kappa)$ by unloading processes allows for its history. It is required to perform the analyses of bent reinforced concrete rod constructions which undergo the static loading. The occurring plasticisation and damage processes by unloading should be estimated experimentally. The knowledge of $\boldsymbol{M}(\kappa)$ while unloading is indispensable not only for the calculation of the deflection of elements with changing loads, but also for the determination of the force redistribution in the structures statically indeterminable. In the first part of the paper a theoretical analysis of the continual relationship moment-curvature $\boldsymbol{M}(\kappa)$ has been carried out. In the second part, it was possible to determine the values of the unknown model functions on the basis of the results of the experimental investigation.

2. Relationship M(κ) in the process of reproducible loads

2.1. General process description

Any history of cross section loading can be presented as a sequence of curvature quantities (measure) $\{\kappa_i\} = \{\kappa_0 = 0, \kappa_1, \kappa_2, \ldots \kappa_n\}$ and corresponding moment quantities $\{M_i\}$ determining points of change in load direction. Separated by two subsequent curvature quantities, relationship course $\boldsymbol{M} - \kappa$ will be called a 'branch' of the loading process. To simplify the notation, for every branch we will accept local coordinate system of coefficients κ'_i, M'_i oriented according to curvature direction change (see Fig. 1). Relationships occur between the global and local coordinate systems:

$\kappa = \kappa_{i-1} \pm \kappa'_i \Rightarrow M = M_{i-1} \pm M'_i$

The plus occurs for loading branches and minus for unloading ones.

2.2. Cross section model

Degradation of cross section stiffness is the result of damage and plasticisation processes. Theoretical basis for such a model has been described by Goszczyński S. [9]. Unified description of processes taking place in reinforced concrete beam cross section can be obtained with the use of elastic work probability, plastic work probability and damage relating to cross section stiffness. During loading, the degree (probability) of cross section plasticising and damage rises, whereas, for the sake of simplification, it can be assumed that these processes are independent from each other. During unloading, stresses of opposing signs additionally degrading stiffness appear in the part of the plasticised cross section zone. In this case, too, independence of the plasticising and damage processes can be postulated but the probabilities determining advancement of these processes are conditional probabilities dependants on the degree of the cross section plasticising during loading. Under the assumptions, function $\boldsymbol{M'}_i(\kappa'_i)$ for 'i' branch can be described by the formula

$$\boldsymbol{M'}_i = (B_o\, \kappa'_i\, \boldsymbol{E}_i + \boldsymbol{P}_i\, \boldsymbol{M}_{p\,i})\, \boldsymbol{N}_{p\,i}\, \boldsymbol{N}_{w\,i} \tag{2.1}$$

where:

B_o – cross section stiffness in the phase Ia,
i – number of load interval,
κ'_i – absolute curvature increase quantity,
$\boldsymbol{E}_i$, $\boldsymbol{P}_i$ – probability functions of elastic and plastic work,
$\boldsymbol{M}_{p\,i}$ – function of moment quantity carried by plasticised cross section,
$\boldsymbol{N}_{p\,i} = N_{p1} - \boldsymbol{R}_{p\,i}$ – probability functions of non-damaging by loading,
$\boldsymbol{N}_{w\,i} = N_{w1} - \boldsymbol{R}_{w\,i}$ – probability functions of non-damaging the cross section by loading,
$\boldsymbol{R}_{p\,i}$ – probability functions of damaging by loading,
$\boldsymbol{R}_{w\,i}$ – probability functions of damaging the cross section by loading.

Since all functions marked with bold italics depend on curvature increase, absolute variables have been omitted in notations.

Two definitions of stiffness are used: secant stiffness marked as $\boldsymbol{B}$ and called shortly "stiffness" with its final value B_i and "instantaneous stiffness" marked as $\boldsymbol{B}(\kappa')$, and its initial value as B_{oi}.

Undamaged part of the cross section works partly elastically and in the rest part plastically, thus at every moment we have

$$\boldsymbol{E}_i + \boldsymbol{P}_i = 1. \tag{2.2}$$

After changing load direction, initially, the whole of undamaged cross section works elastically – the process of plasticising is fully and immediately reciprocal. It causes instantaneous stiffness jump. Initial stiffness of subsequent unload branches is bigger than final stiffness of the previous load branch. During the next loading, the previous state of plasticising the cross section is restored at the moment of exceeding the maximum curvature, causing visible decrease of instantaneous stiffness.

Increase of the stiffness observed during the investigation in the initial intervals of the branch, especially with big load changes, can be explained by pseudo-regeneration of cross section. Since damage relates primarily to concrete being stretched, after changing the direction of load micro cracks and even cracks close. The concrete in these areas starts conveying compression. Thus 'empowering' the cross section will be obtained, assuming reciprocity of damage processes. In contrast to plasticising processes, regeneration is not complete (it occurs with probability $p = 0$). Taking into consideration regeneration it leads to characteristic, 'S' shape of the branch. Such phenomenon has not been observed yet in the results.

During loading, the initial function quantity $\boldsymbol{N}_w$ i equals final quantity from the proceeding unloading branch. If there is no cross section regeneration, the function remains constant for the whole branch, otherwise, it is an ascending function. Function $\boldsymbol{N}_p$ i has analogical features while unloading.

The process of loading proceeds in time causing rheological effects: creeping and relaxation decreasing stiffness. Changeability of load, even low cycling, increases these effects. During short-lastinginvestigation, "early rheologism" with big, but quickly descending velocity of deformation growth plays an important role. This means decrease of stress. It leads to, after a short period of time or after some teen cycles, stability of the branch. In the presented model describing the process of instantaneous loads, time-independent influence of early rheological phenomena is treated as damage.

In case of monotonic probability function, certain branches are also monotonic functions and theoretical relationship $\boldsymbol{M} - \kappa$ presented in Fig. 2. It indicates representation possibility of real proceeding of this relationship.

2.3. The first branch – loading

If we neglect the results of internal stress caused by concrete ageing, the process of constantly growing load is characterised by function constancy $\boldsymbol{N}_{w1} = \boldsymbol{N}_{w0} = 1$ and initial function values: $\boldsymbol{N}_{p1}(0) = \mathrm{N}_{p0} = 1$, $\boldsymbol{P}_1(0) = 0$ ***and*** $\boldsymbol{E}_1(0) = \mathrm{E}_0 = 1$. Yet the differences observed in the research (e.g. [10], [11]) of initial beam stiffness indicate that these phenomena have significant influence and therefore should be taken into account. In the form of the presented model it means that initial values of the functions discussed do not follow previously described relationships and there is $\mathrm{N}_{p0} < 1$, $\mathrm{N}_{w0} < 1$. It would be possible to observe the occurrence of the initial plasticising in case of asymmetric reinforcement as the difference in initial stiffness for bending moments of opposing signs. So far no data is available which would confirm the occurrence of this phenomenon.

Plasticising process intensity $\boldsymbol{f}_{P}$, according to postulated function characteristics must be a constant function. With cross section deformation κ we will obtain plasticising probability

$$\boldsymbol{P}_1 = \mathrm{P}_0 + \int_0^{\kappa} \boldsymbol{f}_{P1}(\boldsymbol{u})\, \mathrm{d}u = \mathrm{P}_0 + \Delta\boldsymbol{P}_1 \approx \Delta\boldsymbol{P}_1, \tag{2.3}$$

and quantity of moment conveyed by plasticised cross section zone

$$\boldsymbol{P}_1 \boldsymbol{M}_{p1} = \int_0^{\kappa} \boldsymbol{u}\, \mathrm{B}_o \boldsymbol{f}_{P1}(\boldsymbol{u})\, \mathrm{d}u = \boldsymbol{P}_1\, \varpi_1\, \mathrm{B}_o\, \kappa, \tag{2.4}$$

where: $0 < \varpi < 1$ – determines the ratio of moment conveyed by plasticised cross section and elastically working cross section.

$\boldsymbol{f}_{P1}$ – intensity of plasicising process with deformation κ. applying the introduced notation, we will obtain the quantity of the moment equal

$$\boldsymbol{M} = \mathrm{B}_o\, \kappa\, [1 - \boldsymbol{P}_1 (1 - \varpi_1)]\, (\mathrm{N}_{p0} - \Delta\boldsymbol{R}_{p1})\, \mathrm{Nw}_1, \tag{2.5}$$

where $\Delta\boldsymbol{R}_{p1}$ – increase of damage During loading.

Characteristics of cross section work in the initial stage of loading ($\kappa \to 0$) will be obtained from the equation Eq. (2.5). The concrete which is properly cared for has little contraction and in most cases with a small error the following equation can be accepted $\boldsymbol{N}_{w1} = \mathrm{N}_{w0} = 1$. After omitting small higher dimensions the equation has the form

$$\boldsymbol{M} = \mathrm{N}_{p0}\, \mathrm{B}_o\, \kappa = \mathrm{B}_{o1}\, \kappa\, . \tag{2.6}$$

Initial stiffness B_{o1} equals tangent of tangent slope at the point $\kappa = 0$ and can be determined directly from the experiment results. This enables calculating initial non-damage of cross section

$$\boldsymbol{N}_{p1}(0) = \mathrm{B}_{o1} / \mathrm{B}_o = \mathrm{N}_{p0}. \tag{2.7}$$

Instantaneous stiffness of the first branch can be obtained by differentiation of the equation Eq. (2.1)

$$\boldsymbol{B}(\hat{e}) = \left(\mathrm{B}_0\, \boldsymbol{E} + \mathrm{B}_0\, \hat{e} \left| \frac{\partial \boldsymbol{E}}{\partial \hat{e}} \right| + \left| \frac{\partial \boldsymbol{P}}{\partial \hat{e}} \right| \boldsymbol{M}_p + \left| \frac{\partial \boldsymbol{M}_p}{\partial \hat{e}} \right| \boldsymbol{P} \right) \boldsymbol{N}_p \boldsymbol{N}_w +$$

$$+ \left(\mathrm{B}_0\, \hat{e}\, \boldsymbol{E} + \boldsymbol{P} \boldsymbol{M}_p \right) \left(\left| \frac{\partial \boldsymbol{N}_p}{\partial \hat{e}} \right| \boldsymbol{N}_w + \left| \frac{\partial \boldsymbol{N}_w}{\partial \hat{e}} \right| \boldsymbol{N}_p \right) \tag{2.8}$$

It is impossible to determine functions in the equation Eq. (2.1) or Eq. (2.5), on the basis of the empirical function of the first branch.

2.4. The second branch – unloading

Equation of the moments for this branch has the form

$$\boldsymbol{M}'_2 = \mathrm{B}_o\, \kappa'_2\, [1 - \boldsymbol{P}_2 (1 - \varpi_2)]\, \mathrm{N}_{p2}\, (\mathrm{N}_{w1} - \Delta\boldsymbol{R}_{w2}), \tag{2.9}$$

where: $\mathrm{N}_{p2} = \mathrm{N}_{p1}$ – probability of non-damage by loading at the moment of unloading,

$\boldsymbol{P}_2$ – plasticising probability during unloading,

$\Delta\boldsymbol{R}_{w2}$ – increase of damage probability during unloading,

ϖ_2 – determines the ratio between the moment conveyed by the cross section plasticised by unloading and the cross section working elastically.

Next we will use the equality of certain initial probability of non-damage, constant for the given branch and final for the proceeding branch.

Initial stiffness for this branch determined from the equation Eq. (2.9), equals

$$B_{o2} = B_o\, N_{p2}\, N_{w1} \tag{2.10}$$

which enables to calculate the value of non-damage function at this point of the process on the basis of measurement results

$$N_{p2} = \boldsymbol{N_p}(\kappa_1) = B_{o2} / (B_o\, N_{w1})$$

$$\Delta R_{p1} = 1 - N_{p2} \tag{2.11}$$

Stretching stress has the biggest influence on stiffness degradation, including during unloading. Since the stretched concrete presents features of a fragile body, the probability of plasticising effects will be small. This means that at least at the first approximation it can be assumed as zero. It allows to simplify the equation Eq. (2.9) as

$$\boldsymbol{M'_2} = B_o\, \kappa'_2\, N_{p2}\, (N_{w1} - \Delta \boldsymbol{R_{w2}}) \tag{2.12}$$

describing non-linear, elastic - fragile cross section model. With increasing unload, the probability of damage and stiffness decrease. Its final quantity can be written as

$$\boldsymbol{B_2} = B_o\, N_{p2}\, (N_{w1} - \Delta \boldsymbol{R_{w2}}) = B_o\, N_{p2}\, N_{w2} \tag{2.13}$$

Comparing to the final stiffness of the first branch it is bigger since damage probability ΔR_{w2} is the fraction of plasticity probability P_1.

Knowing the initial stiffness and the coefficient of branch ending enables us to calculate the increase of damage ΔR_{w2}:

$$M'_2 = B_o\, \kappa'_2\, N_{p2}\, N_{w2} \Rightarrow N_{w2} =$$

$$= \frac{M'_2}{B_o\, \hat{e}'_2\, N_{p2}} \Rightarrow \Delta R_{w2} = N_{w2} - N_{w1} \tag{2.14}$$

2.5. The third branch (loading) – deformation hysteresis loop

The subsequent loading, with deformation not exceeding κ_1, gives linear function

$$\boldsymbol{M'_3} = B_2\, \kappa'_3 = B_{o3}\, \kappa'_3 \tag{2.15}$$

because subsequent loading does not cause additional damage of the cross section. Also following unloading and loading in the range $<\kappa_1, \kappa_2>$ do not change stiffness; we have a stable loop (unloading – loading) of deformation hysteresis. In the case of fragile material, the loop is reduced to the interval of a straight line.

In reality, obtaining the curvature κ_1 is caused by a smaller moment than M_1 since the influence of rheological phenomena appears. The equation of the third branch, closing the first loop, allowing for rheological effects, assuming $N_{w3} = N_{w2}$, can be presented as follows

$$\boldsymbol{M'_3} = B_o\, \kappa'_3\, (N_{p2} - \Delta \boldsymbol{R_{p3}})\, N_{w3} \tag{2.16}$$

where initial stiffness equals zero:

$$B_{o3} = B_o\, N_{p2}\, N_{w3} \tag{2.17}$$

Increase of damage for $\kappa'_3 = \kappa'_2$ can be calculated analogically as for the second branch:

$$M'_3 = B_o\, \kappa'_3\, N_{p3}\, N_{w3} \Rightarrow N_{p3}$$

$$= \frac{M_3}{B_o \hat{e}'_3\, N_{w3}} \Rightarrow \Delta R_{p3} = N_{p2} - N_{p3} \tag{2.18}$$

To determine the quantity of damage for this branch there can be used difference between moments:

$$\Delta M'_{1,3} = M_1 - M_3 = B_o (\kappa'_1 - \kappa'_2)\, \Delta R_{p3}\, N_{w3} \tag{2.19}$$

While analysing deformation hysteresis loop, it is more convenient to use variables κ'_p for the increasing load and κ'_w for the decreasing one. Loop equations can be generalised for the subsequent m = 1, 2, …, n cycle deformation loops in this interval:

– for unloading branches

$$\boldsymbol{M'_i} = B_o\, \kappa'_w\, N_{p\,i}\, (N_{w\,i-1} - \Delta \boldsymbol{R_{w\,i}}) \tag{2.20}$$

– for loading branches

$$\boldsymbol{M'_{i+1}} = B_o\, \kappa'_p\, (N_{p\,i} - \Delta \boldsymbol{R_{p\,i+1}})\, N_{w\,i+1} \tag{2.21}$$

where: i = 2 m.

Initial stiffness of these branches, having allowed for the final of the relevant process at the moment of load direction change, equal respectively:

$$B_{o\,i} = B_o\, N_{p\,i-1}\, N_{w\,i-1} = B_o\, N_{p\,i}\, N_{w\,i-1}$$

$$B_{o\,i+1} = B_o\, N_{p\,i}\, N_{w\,i} = B_o\, N_{p\,i}\, N_{w\,i+1} \tag{2.22}$$

The stiffness changes reflect accumulating damages and creeping. They occure as a result of earlier loading. They result from characteristics of loop slope decreasing with the increase of the cycle number.

Increase of the moment for the lower deformation limit (even branch number) in the following cycle ($\Delta\kappa = \kappa_1 - \kappa_2$) amounts to

$$\Delta M'_{i,i-2} = M_i - M_{i-2} = B_o\, \Delta\kappa\, (N_{p\,i-1}\, N_{w\,i-1} - N_{p\,i-1}\, N_{w\,i})$$

$$= - B_o\, \Delta\kappa\, N_{p\,i}\, \Delta R_{w\,i},$$

and for the upper one

$$\Delta M'_{i+1,i-1} = - B_o\, \Delta\kappa\, \Delta R_{p\,i+1}\, N_{w\,i+1} \tag{2.23}$$

This allows to calculate the range of moment changes after stabilising the deformation hysteresis loop (Fig. 3):

$$M_{min} = M_1 - B_o\, \Delta\kappa\, N_{p2}\, N_{w1} + B_o\, \Delta\kappa \sum_{1}^{m} N_{p\,2m}\, \Delta R_{w\,2m} \tag{2.24}$$

$$M_{max} = M_1 - B_o\, \Delta\kappa \sum_{1}^{m} \Delta R_{p\,2m+1}\, N_{w\,2m} \tag{2.25}$$

Considering small increases of deformation, equations Eq. (2.24), (2.25) can be simplified as:

$$M_{min} \approx M_1 - B_o\, \Delta\kappa\, N_{p2}\, N_{w1} +$$
$$+ B_o\, \Delta\kappa \left(N_{p2} - \sum_{1}^{m} \Delta R_{p\,2m+1}\right) \sum_{1}^{m} \Delta R_{w\,2m} =$$
$$= M_1 - B_o\, \Delta\kappa\, N_{p\,2}\, N_{w\,1} + B_o\, \Delta\kappa\, [N_{p\,2} - \boldsymbol{R}_p(\Delta\kappa)]\, \boldsymbol{R}_w(\Delta\kappa),$$
$$M_{max} \approx M_1 - B_o\, \Delta\kappa \left(N_{w1} - \sum_{1}^{m} \Delta R_{w\,2m}\right) =$$
$$= M_1 - B_o\, \Delta\kappa\, [N_{w1} - \boldsymbol{R}_w(\Delta\kappa)] \tag{2.26}$$

enabling determination of damage quantity on the basis of measurement.

2.6. The third and the subsequent branches – moment hysteresis loop

Increasing load in the third branch until $M_3 = M_1$ (Fig. 4) causes simultaneously an increase in the curvature until the quantity κ_3. As the consequence we observe not only an increase of damage probability but also an increase of plasticising. The subsequent unloading, as a result of increase in the plasticising degree, increases damage caused by internal stress, as well. There are added rheological effects, too. The undergoing changes in the subsequent cycles {i = 2, 4, …, 2n} can be noted as:

$$\begin{aligned} \Delta P_{i+1} &= \boldsymbol{P}(\kappa_{i+1}) - \boldsymbol{P}(\kappa_{i-1}), \\ \Delta R_{p\,i+1} &= \boldsymbol{R}_p(\kappa_{i+1}) - \boldsymbol{R}_p(\kappa_{i-1}), \\ \Delta R_{w\,i} &= \boldsymbol{R}_w(\kappa_i) - \boldsymbol{R}_w(\kappa_{i-2}), \end{aligned} \tag{2.27}$$

where there have been presented together: damage increase, curvature increase and rheological effects.

If, in the process of cycle loading, the curvature sequence $\{\kappa_{2m+1}\}$ has a finite limit $\kappa_{max} = \kappa_{2n+1}$, then the hysteresis loop is stable in the deformation interval $<\kappa_{min} = \lim\{\kappa_{2n}\}, \kappa_{max}>$. Respective probabilities determining the state of the cross section amount to:

$$\boldsymbol{P}_m = \boldsymbol{P}_1 + \sum_{j=1}^{n} \Delta P_{2j+1} = \boldsymbol{P}(\kappa_{2m+1}) = \boldsymbol{P}(\kappa_{max}),$$

$$R_{p\,m} = R_{p\,1} + \sum_{j=1}^{n} \Delta R_{p\,2j+1} = \boldsymbol{R}_p(\kappa_{max}), \tag{2.28}$$

$$R_{w\,m-1} = \sum_{j=1}^{n} \Delta R_{w\,2j} = \boldsymbol{R}_w(\kappa_{max} - \kappa_{min}),$$

$$N_{p\,m} = 1 - R_{p\,m} = \boldsymbol{N}_p(\kappa_{max}),$$
$$N_{w\,m} = 1 - R_{w\,m} = \boldsymbol{N}_w(\kappa_{max} - \kappa_{min}),$$

and they are the same as in the deformation hysteresis loop $<\kappa_{min}, \kappa_{max}>$.

3. Determining model functions $\boldsymbol{N_p}$, $\boldsymbol{N_w}$, $\boldsymbol{E}$, $\boldsymbol{P}$, $\boldsymbol{M_p}$

The relationships between the instantaneous and section stiffness with cycle loads described in point 2 allow to determine damage increase ΔR_p and ΔR_w. As a result, values N_p, N_w model damage functions at the points of changing load direction become known. At the point of initiating the first level of unload cycles, with known $N_{p\,1}$, $N_{w\,1}$ we have the equations Eq. (2.1) and Eq. (2.2). Calculating the values E_1, P_1, $M_{p\,1}$ is still impossible. Similarly, it is impossible at the initial points of the higher unload levels.

The missing equation is the result of instantaneous stiffness quantity at the point of load direction change. At the point of initiating cycle loads for any 'k' level of moment hysteresis loop we have:

$$\left|\frac{\partial \boldsymbol{M}}{\partial \hat{e}}\right|_{\hat{e}_k} = \left(B_0 E_k + B_0\, \hat{e}_k \left|\frac{\partial \boldsymbol{E}}{\partial \hat{e}}\right|_{\hat{e}_k} + \left|\frac{\partial \boldsymbol{P}}{\partial \hat{e}}\right|_{\hat{e}_k} M_{p\,k} + \left|\frac{\partial \boldsymbol{M}_p}{\partial \hat{e}}\right|_{\hat{e}_k} P_k \right) N_{p\,k}\, N_{w\,k-1} +$$
$$+ \left(B_0\, \hat{e}_k E_k + P_k\, M_{p\,k}\right) \left(\left|\frac{\partial \boldsymbol{N}_p}{\partial \hat{e}}\right|_{\hat{e}_k} N_{w\,k-1} + \left|\frac{\partial \boldsymbol{N}_w}{\partial \hat{e}}\right|_{\hat{e}_k} \boldsymbol{N}_{p\,k} \right) \tag{3.1}$$

Little section of empirical relationship $\boldsymbol{M}(\kappa)$ just before the first point of initiating the loop is approximated with a parabola

$$\boldsymbol{M}(\kappa) = a\,\kappa^2 + b\,\kappa + c \tag{3.2}$$

for which the coefficients a, b, c are determined by the method of the least quares.

As a result, the value of the left side of the equation (3.1) becomes known.

At this point, considering the equations Eq. (2.1), (2.2) and (3.1) we have both unknown E_k, P_k, $M_{p\,k}$ and quantities of model function derivatives $\boldsymbol{E}$, $\boldsymbol{P}$, $\boldsymbol{M}_p$ as well. For initial points of subsequent load levels derivative values $\left|\frac{\partial \boldsymbol{E}}{\partial \hat{e}}\right|_{\hat{e}_k}$ can be determined on the basis of parabola lead through values E_{k-1}, E_k, E_{k+1}.

The same method is used for derivatives $\left|\frac{\partial \boldsymbol{P}}{\partial \hat{e}}\right|_{\hat{e}_k}$, $\left|\frac{\partial \boldsymbol{M}_p}{\partial \hat{e}}\right|_{\hat{e}_k}$. Equations (2.1), (2.2) and (3.1) noted for the initial point of 'k' unload level contains nine unknowns E_{k-1}, E_k, E_{k+1}, P_{k-1}, P_k, P_{k+1}, $M_{p\,k-1}$, $M_{p\,k}$,

$M_{p\ k+1}$. Equations (2.1), (2.2) and (3.1) for all initial points of loading levels form a system of the same number of equations as the unknowns. Due to non-linearity of these equations model function values $E_1, \ldots, E_{k-1}, E_k, E_{k-1}, \ldots, E_n, P_1, \ldots, P_{k-1}, P_k, P_{k-1}, \ldots, P_n, M_{p1}, \ldots, M_{p\,k-1}, M_{p\,k}, M_{p\,k-1}, \ldots, M_n$, for points initiating cycles they are determined numerically with the use of Newton method [12].

4. Experimental research of three span reinforced concrete beams.

It is characteristic of the publicised research results of continuous reinforced concrete beams [13-15] that the research programs are limited to the issues of interest for the different researchers.

Experimental discovery of the cross section and an element work with temporary load (without dynamic effects) considering the unloads have been carried out on three span reinforced concrete beams. It was necessary to carry out the experiment in Independent School of Reinforcement Construction and Industrial Building of Technical University of Kielce due to lack of research results allowing to determine model function values of relationship M(κ).

The static scheme of three span continuous beam has been accepted (Fig. 5), with rectangular cross section 140 x 250 mm, and the length of peripheric spans 1750 mm and midspan 2750 mm. The beams were made of concrete with average resistance to compression about 22 MPa. Percentage of reinforcement with bars # 12 from steel 34 GS amounted to 1.07%.

Two beams have been tested '1' and '2'. Two focused forces applied in the midspan formed the load. The program of static load (controlling of focused force P) were done according to scheme $P = 0 \rightarrow P_1 \leftrightarrow 0 \rightarrow P_2 \leftrightarrow 0 \rightarrow P_3$ ..., where $P_i > P_{i-1}$, P_U – damaging force. The number of loading and unloading cycles from the given force level P was dependant on measurable hysteresis loop stability force - deflection.

During measurement program for the tested elements were registered:

- the quantity of the applied load P and all the reactions on the supports: A, B, C, D (see Fig. 6)
- deflection in the determined points of the beams (see Fig. 6)
- deformation of upper and lower fibres on both sides of the beam (see Fig. 6)
- location, height and opening of cracks.

Testing of continuous beams was carried out on a measuring stand.

For the measurements the following devices were used:

- box dynamometers (range 25 – 160 kN) for measuring applied load and bearing reaction; estimated on the basis of calibration of max measured force error for the different dynamometers from 0.19 – 0.44 kN,
- induction sensors by PELTRON (range 50 – 100 mm) for measuring deflection; accuracy 1.25×10^{-5} m,
- extensometer type MAYES (range 200 mm) for measuring deformation; accuracy 0.8×10^{-5}
- Brinell microscope (enlargement 24 times) for measuring crack opening.

The quantity of the applied load, bearing reaction and deflection, were registered with thirty two channel analogue – digital converters mounted into an IBM PC. This allowed to gather a large number of measuring points. The converter with jump load increase P for the assumed level, sampled automatically five times with the same load kept, every 20 seconds for 5 minutes.

5. Empirical hysteresis loops and their analysis.

For the interval of sheer bending in the span ***M***(κ) has been calculated on the basis of the measured deformations and deflections. There the constant intensity of longitudinal reinforcement exists simultaneously. The calculated relationships ***M***(κ) differed maximally by ~12%. The confirmed reliability of the results computed from automatic measurement of deflection allowed to base the analysis exclusively on them.

The experimental curvature was calculated based on the formula:

$$\kappa = \mathbf{8}\,\Delta f / l^2 \qquad (5.1)$$

where: Δf – the increase in the deflection in the middle of the segment l in relationship to its ends l – the length of the segment of sheer bending. Chosen as an example two series of moment loops M(κ) for one of the beams have been shown in (Fig. 7). They are both with the ranges respectively ~26 – 0 kNm and ~29 – 0 kNm. For each series there are from several to several teens unloading and loading branches lying close to each other. Due to the measurement carried out only in a few points of the branch, linking results are presented as broken line. Observed subsequent branch overlapping, overlaying, crossing may be explained both by the character of the phenomenon and the measuring accuracy.

The obtained empirical points with coefficients (M, κ) in the Cartesian system are loaded with errors ΔM and Δκ, resulting from the accuracy of measured bearing reactions, applied load and deflection measurement. On the basis of measuring equipment calibration [15] maximal value of span moment error ΔM_g = 1.23 kNm and curvature error $\Delta\kappa_g$ = 0.0002 1/m have been estimated.

In order to determine the shape of 'm' loop from the sequence, the following analysis has been carried out. For any empirical point 'L' with coefficients (κ_L, M_L) lying on the load or unload curve (Fig. 8) vertical distances $-\Delta M_L$ and horizontal ones $\Delta\kappa_L$ from the straight line run through the upper and lower vertex of the loop have been calculated. It arises from the statistic comparison of the sets of the vertical $\{\Delta M_L\}$ and horizontal $\{\Delta\kappa_L\}$ distances to maximal measuring errors of the moment ΔM_g = 1.23 kNm and curvature $\Delta\kappa_g$ that the loop M(κ) can be linearised in the loading and unloading process.

The sequence of hysteresis loop has been limited by the contour of a quadrangle as in Fig. 9.

Unloads and repeated loads cause additional damage process which is proved by positive sums of temporary and permanent curvature increase. Having formed the sequence of hysteresis loops from any point 'k' on the original curve $M^0(\kappa)$, we do not get to the lengthening of the curve (Fig. 9).

6. Probability values *E, P, N_p, N_w* and moment quantity *M_p* in the initial unload points of 'κ' load level with estimation of errors Δ*E*, Δ*P*, Δ*N_p*, Δ*N_w*, Δ*M_p*.

The determined relationship *M*(κ) for the middle beam interval of 1000 mm length served for calculating values of model functions ***E, P, N_p, N_w, M_p***. Relationship *M*(κ) is loaded with error of bending moment ΔM_g resulting from accuracy of bearing reaction measurement [16]. Curvature error $\Delta\kappa_g$ has been omitted considering it a small one in comparison with maximal calculated curvature.

If the calculated quantity depends on many supporting quantities, there is a very small probability of occurrence of an event in which errors of all supporting quantities will take extreme values and unfavourable set of signs. The best results can be obtained if we treat the measuring results together with its error as a random variable. In this case, in order to estimate its error, we can use radical form variation called an absolute probable average error [17].

If for the random variable

$$Y = f(X_1, X_1, \dots, X_n) \tag{6.1}$$

non-linearity of the function f has little importance and coefficients of variable changing X_i are not too big, then approximation of the variable variation Y in the most general case is as follows

$$\mathrm{Var}[Y] \approx \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial f}{\partial X_i}\frac{\partial f}{\partial X_j}\mathrm{Cov}[X_i, X_j] \tag{6.2}$$

where: Cov[X_i , X_j] means covariantion of variables X_i, X_j.

Due to independence of variables X_i covariantion are zero and then

$$\mathrm{Var}[Y] \approx \sum_{i=1}^{n}\left(\frac{\partial f}{\partial X_i}\right)^2 \mathrm{Var}[X_i]. \tag{6.3}$$

In the analysis, it has been assumed that function values ***E, P, N_p, N_w, M_p*** are loaded with errors because of bending moment error ΔM_{gmax}. It arises from the analysis of bearing reaction [16] that for any point of the empirical relationship M(κ) the calculated ordinate is the expected value of the bending moment. It follows the normal distribution with standard deviation equalling one third ΔM_{gmax}. Estimating the errors Δ***E***, Δ***P***, Δ***N_p***, Δ***N_w***, Δ***M_p*** we calculate the radicals from variation which is standard deviation.

Obtaining directly the analytical formulas of errors Δ***E***, Δ***P***, Δ***N_p***, Δ***N_w***, Δ***M_p*** with the use of (6.2) or (6.3) is impossible. The most accurate solution is to make use of the probabilistic nature of errors Δ***E***, Δ***P***, Δ***N_p***, Δ***N_w***, Δ***M_p*** stemming from the bending moment error ΔM_{gmax}. For this purpose, its necessary to sample the value of the moment random variable for subsequent measuring points of the relationship M(κ) with the application of elemental simulation. In an artificial manner for the relevant measuring points (κ, M) the values of random variable M have been generated. Variability according to normal distribution with expected value equal to empirical bending moment value in the given point and standard deviation of one third value of ΔM_{gmax} have been assumed. The normal distribution was generated with Monte Carlo simulation method [18].

While estimating error values Δ***E***, Δ***P***, Δ***N_p***, Δ***N_w***, Δ***M_p*** thirty simulations have been carried out. They yield a set of thirty curvatures M(κ) for every beam. Numerically determined thirty-element sets with Newton method ***E***, ***P***, ***N_p***, ***N_w***, ***M_p*** were used for calculating average values and standard deviations. The determined maximal in the observed range of loads relative error values δ***E***, δ***N_p***, δ***N_w***, δ***M_p*** equal respectively 13.1, 6.4, 4.3, 12.8 [%].

The course of the determined model function values ***E***, ***P***, ***N_p***, ***N_w***, ***M_p*** in the function of effort M/M_u (M_u - empirical damage moment) is presented graphically in Fig. 10. M_u is known as a result of the created plastic coupling on the section of sheer bending for both tested beams.

The experimental research presented in this paper with analysis of the obtained results allows us to state the following:

- in the range of loads up to ~0.9 of damage moment the influence of unloads on reinforced concrete beams is small. There is only observed damage increase of cross section stiffness of ~15-18%,
- the process of damage during loading is significant as for effort ~0.9 probability of cross section damage amounts to 59-66%,
- cross section plasticising probability in the researched range increases on average from 40 to 85%,
- in the load range up to ~0.65 M_u a small increase of probability elastic work maximally up to ~5% is observed. This indicates bigger concentration of damage in the plasticised parts of cross section,
- for practical uses hysteresis loop in the changing load process can be linearised; the slope angle of the straight line depends only on the product of values of non-damage probability $N_{p\,k}\,N_{w\,k}$.

The obtained values of the model function reflect quantitative influence of damaging processes and plasticisation on cross section stiffness degradation with the accepted mathematical model (2.1). Physical interpretation of the model functions stems from treating the cross section as a set of elementary sectors behaving elastically, plastically and in a short way for any yield point and deformation point being random variables.

7. Summary

The relationship M(κ) valid not only for the process of increasing load but also unloading is suggested in this paper. It is continual, non-linear limited to five model functions $\boldsymbol{E}$, $\boldsymbol{P}$, $\boldsymbol{N}_p$, $\boldsymbol{N}_w$, $\boldsymbol{M}_p$. reflecting its real character. Physical interpretation of model functions stems from the following assumptions:

- the cross section is treated as a set of material points with random mechanical features,
- elementary sectors behave in an elastic, plastic and in a short way, with yield point and deformation point being random variables,
- the features on the level of the cross section are represented by respective values of the five functions or their proper combinations in the accepted mathematical model.

The model generalises theoretical descriptions of relationships M(κ) in the processes of loading and unloading. The model functions reflect mechanisms related to micro and macro cracking, plasticising and self-stress. The simplicity and clarity is the advantage of so formed relationship. Thus, its not necessary to analyse the relationships between deformation, normal stress or adhesion stress between two neighbouring cracks.

The values of the unknown model functions have been determined on the basis of the small database of experiment results of M(κ) allowing for low-cycle loading. This limits the validity of the model to the specific cross section shape with given percentage content of reinforced steel both for stretching and compressing, its kind and concrete characteristics.

The small number of the tested beams together with the existing estimated errors $\Delta\boldsymbol{E}$, $\Delta\boldsymbol{P}$, $\Delta\boldsymbol{Np}$, $\Delta\boldsymbol{N}_w$, $\Delta\boldsymbol{M}_p$ about ~13% call for caution in the quantitative evaluation of the individual processes. For practical uses hysteresis loop in the changing load process can be linearised. The slope angle of the straight line depends only on the product of values of non-damage probability $N_{p\,k}\,N_{w\,k}$.

For a better quantitative evaluation of the damage and cross section plasticisation processes it is required to carry out broader experimental research.

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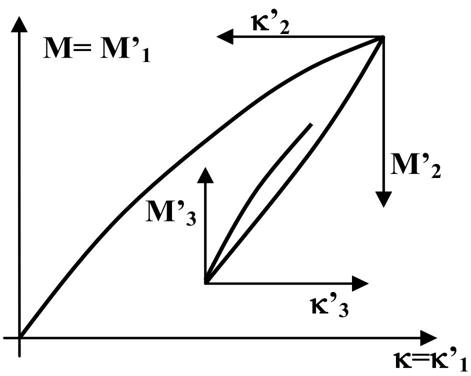

Fig. 1. Coordinate systems of relationship M – κ

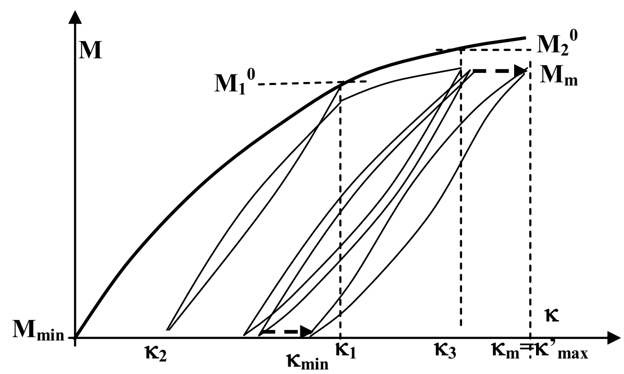

Fig. 2. Scheme relationship M(κ) for changing loads

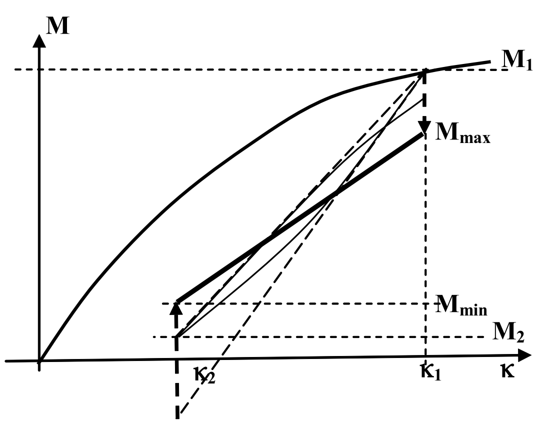

Fig. 3. Deformation hysteresis loop

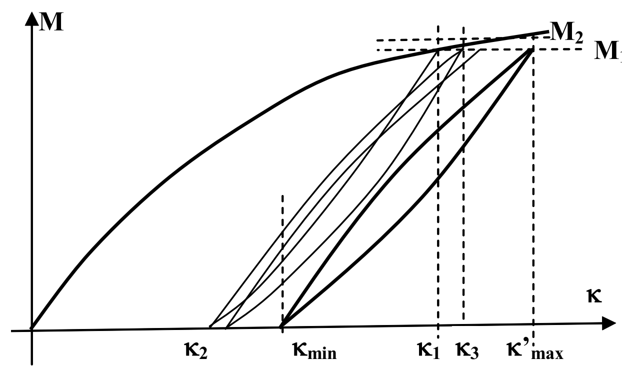

Fig. 4. Moment hysteresis loop

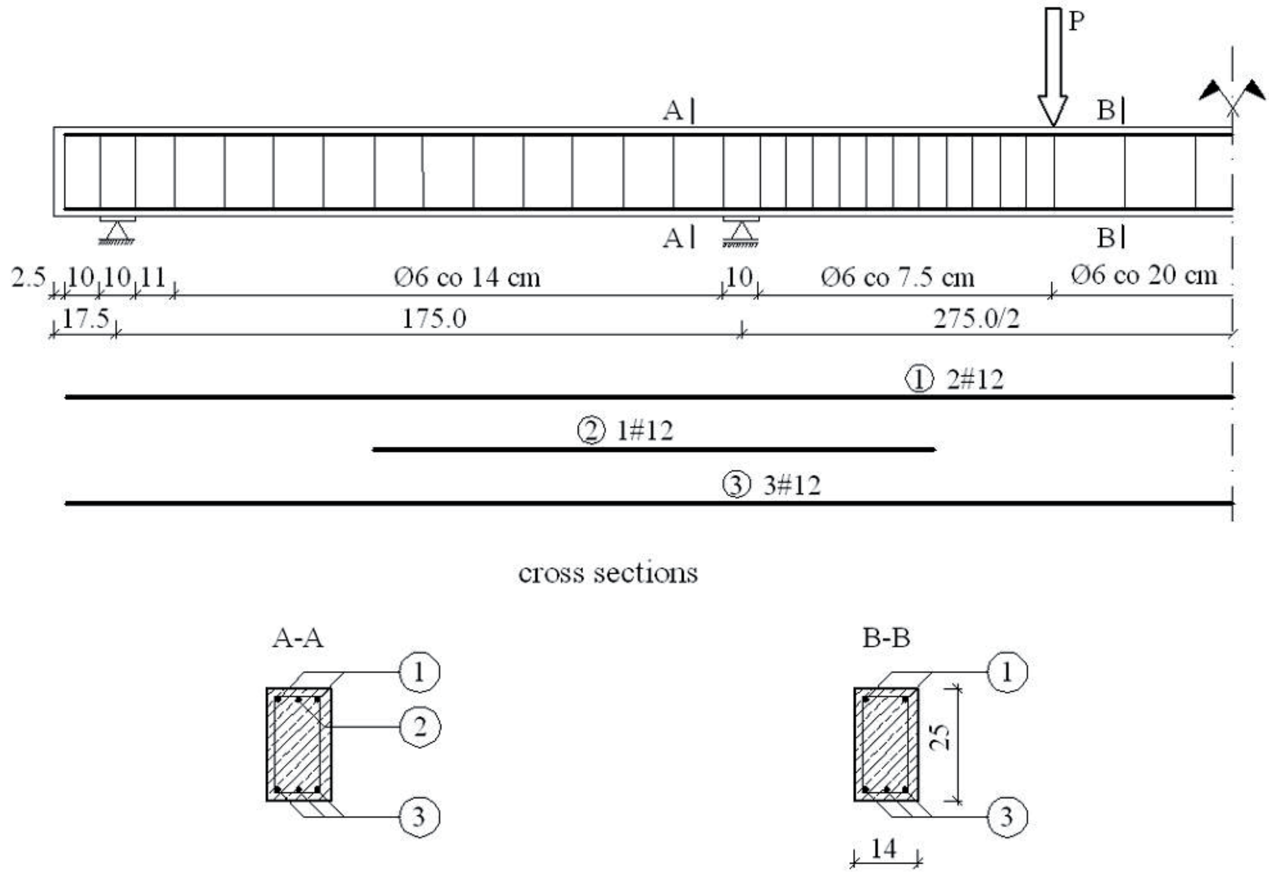

Fig. 5. Static scheme, beam reinforcement

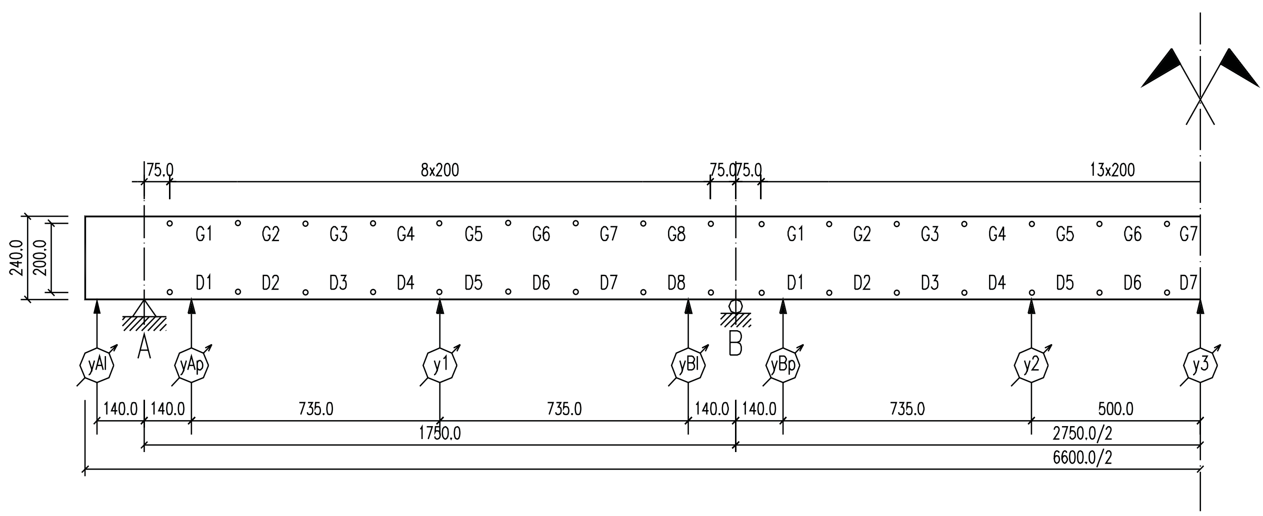

Fig. 6. Distribution of measuring points of deflection and deformation basis

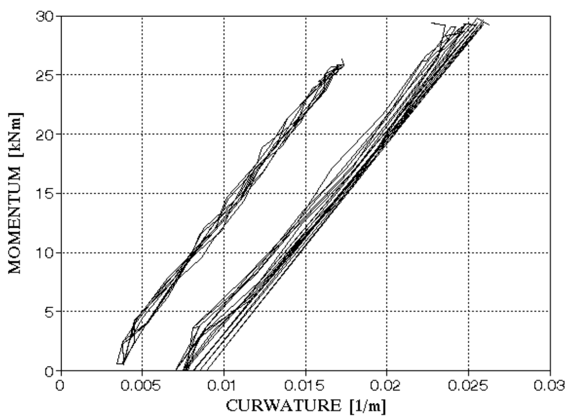

Fig. 7. Example two series of moment loops M(κ)

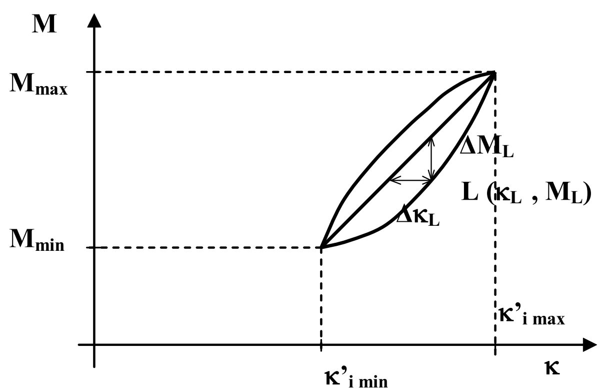

Fig. 8. Distance from the empirical point 'L' of the loop *M*(κ) from the linearised form

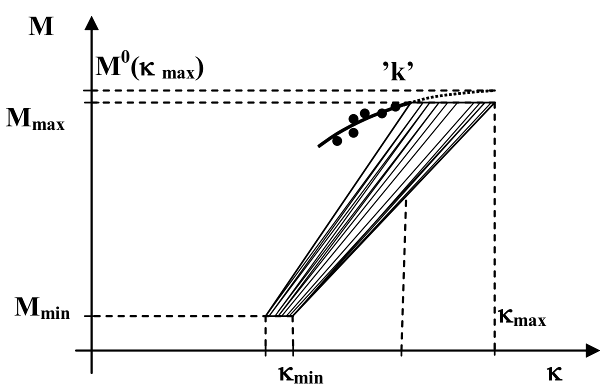

Fig. 9. Linearisation of load and unload branches of relationship *M*(κ)

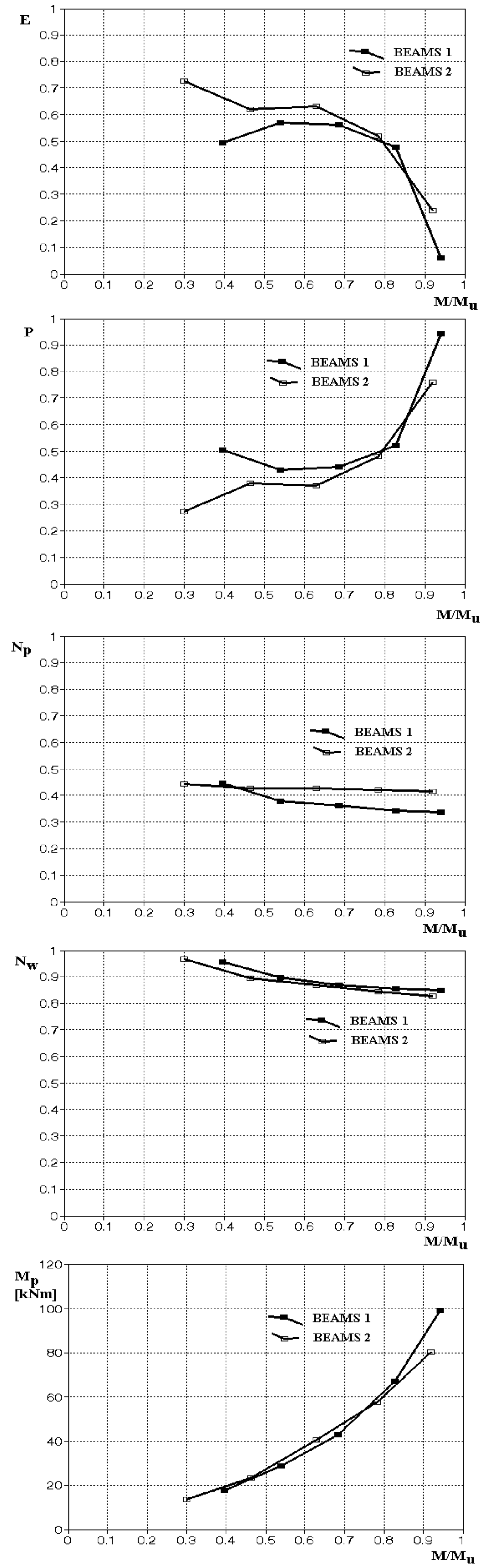

Fig. 10. The course of average values ***E***, ***P***, N_p, N_w, M_p in the function of effort M/M_u

Jacek Śluszrczyk

Zależność M(κ) na podstawie wyników badań belek poddanych obciążeniom zmiennym

1. Wstęp

Przy obliczaniu konstrukcji żelbetowych szczególne znaczenie ma relacja moment - krzywizna $\boldsymbol{M}(\kappa)$. Zależność ta decyduje o dystrybucji sił wewnętrznych w ustrojach statycznie niewyznaczalnych oraz o wielkościach ugięć elementów. W rzeczywistych warunkach eksploatacyjnych, konstrukcja żelbetowa poddawana jest nie tylko obciążeniom narastającym ale i odciążeniom.

W elementach zginanych wykonanie nawet kilku cykli statycznych obciążeń i odciążeń wywołuje przyrost ugięć doraźnych. Budowa jednolitego modelu zależności $\boldsymbol{M}(\kappa)$ opisującego cały zakres od stanu sprężystego do pełnego uplastycznienia z uwzględnieniem obciążeń powtarzalnych (pętli histerezy) wymaga przeprowadzenia odpowiednich badań.

2. Model przekroju

Degradacja sztywności przekroju jest wynikiem procesów niszczenia i uplastycznienia. Jednolity opis procesów zachodzących w przekroju belki żelbetowej można uzyskać wykorzystując prawdopodobieństwa pracy sprężystej, plastycznej oraz niszczenia, odniesionych do sztywności przekroju. Podczas obciążania, rośnie stopień (prawdopodobieństwo) uplastycznienia i zniszczenia przekroju, przy czym można przyjąć dla uproszczenia, że procesy te od siebie nie zależą. W trakcie odciążania, w części uplastycznionej strefy przekroju, powstają naprężenia przeciwnych znaków dodatkowo degradujące sztywność. Również w tym wypadku można postulować niezależność procesów niszczenia i uplastycznienia, ale prawdopodobieństwa określające zaawansowanie tych procesów są prawdopodobieństwami warunkowymi zależnymi od stopnia uplastycznienia przekroju podczas obciążania. Wykorzystując przyjęte założenia, funkcję $\boldsymbol{M'}_i(\kappa_i')$ dla i – tej gałęzi można opisać wzorem

$$\boldsymbol{M'}_i = (B_0\, \kappa'_i\, \boldsymbol{E}_i + \boldsymbol{P}_i\, \boldsymbol{M}_{p\,i})\, \boldsymbol{N}_{p\,i}\, \boldsymbol{N}_{w\,i}, \qquad (1)$$

gdzie:

B_0 – sztywność przekroju w fazie Ia,
i – numer przedziału obciążania,
κ'_i – bezwzględna wielkość przyrostu krzywizny,
$\boldsymbol{E}_i$, $\boldsymbol{P}_i$ – funkcje prawdopodobieństwa pracy sprężystej i plastycznej,
$\boldsymbol{M}_{pi}$ – funkcja wielkości momentu przenoszonego przez przekrój uplastyczniony,
$\boldsymbol{N}_{p\,i} = N_{p\,1} - \boldsymbol{R}_{p\,i}$ – funkcja prawdopodobieństwa niezniszczenia obciążeniem
$\boldsymbol{N}_{w\,i} = N_{w\,1} - \boldsymbol{R}_{w\,i}$ – funkcja prawdopodobieństwa niezniszczenia przekroju odciążeniem
$\boldsymbol{R}_{p\,i}$ – funkcja prawdopodobieństwa zniszczenia obciążeniem
$\boldsymbol{R}_{w\,i}$ – funkcja prawdopodobieństwa zniszczenia przekroju odciążeniem

Ponieważ wszystkie funkcje wyróżnione pogrubioną kursywą zależą od przyrostu krzywizny, w zapisach pominięto wskazanie odciętych.

W przypadku monotonicznych funkcji prawdopodobieństw, poszczególne gałęzie są również funkcjami monotonicznymi i przedstawiony schematycznie na rys. 1 teoretyczny związek $\boldsymbol{M}$-κ wskazuje na możliwość odwzorowania rzeczywistego przebiegu tej zależności.

3. Wyznaczenie wartości funkcji modelowych na podstawie badań doświadczalnych.

Doświadczalne poznanie pracy przekroju i elementu pod obciążeniem doraźnym (bez efektów dynamicznych) z uwzględnieniem odciążeń przeprowadzono na trzech trójprzęsłowych belkach żelbetowych (BELKA1, BELKA 2, BELKA 3). Przebieg wyznaczonych wartości funkcji modelowych $\boldsymbol{E}$, $\boldsymbol{P}$, $\boldsymbol{N}_p$, $\boldsymbol{N}_w$, $\boldsymbol{M}_p$ w funkcji wytężenia $m = M/M_u$ (M_u - empiryczny moment niszczący) przedstawiono graficznie na rys. 10.

4. Podsumowanie

W zakresie obciążeń do ~0.9 momentu niszczącego wpływ odciążeń na zachowanie się belek żelbetowych jest niewielki, obserwuje się jedynie przyrost niszczenia sztywności przekroju o ~15-18%. Proces niszczenia podczas obciążania jest znaczny gdyż dla wytężeń ~0.9 prawdopodobieństwo zniszczenia sztywności przekroju wynosi 59-66%. Prawdopodobieństwo uplastycznienia przekroju osiąga w badanym zakresie obciążeń wartość 65-80%. W zakresie obciążeń do ~0.70 M_u obserwuje się niewielki wzrost prawdopodobieństwa pracy sprężystej o ~0.03-0.08% co wskazuje na większą koncentrację niszczenia w uplastycznionych częściach przekroju.

W zastosowaniach praktycznych pętle histerezy w procesie obciążeń zmiennych można linearyzować. Kąt nachylenia prostej zależy jedynie od iloczynu wartości prawdopodobieństw niezniszczenia $N_{pk} N_{wk}$.