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STIFFNESS OPERATOR FOR BOUNDARY PROBLEMS OF STRUCTURAL ANALYSIS AND SEVERAL NEW VARIATIONAL FORMULATIONS

Abstract

So-called "stiffness operator" for boundary problems of structural analysis is introduced in the distinctive paper. There is a direct analogy between this stiffness operator and corresponding stiffness matrix of the considering structure in terms of finite element method. Namely we can consider stiffness operator as a limiting (continual) generalization of stiffness matrix. The explicit formulation of stiffness operator is presented. Several new variational formulations are discussed as well.

Keywords: stiffness operator, boundary problems, structural analysis, variational formulations

1. Introduction

The finite element method [2, 7] (FEM) has proven to be a versatile method for the simulation of continuous physical systems in many problems of structural analysis. FEM requires a discretization (a mesh) of a domain. This discretization, along with FE formulations, is used to assemble a set of simultaneous equations

$$K\bar{u} = \bar{f} \tag{1.1}$$

which, when solved, provide an approximate solution of the system under consideration. Here K is a stiffness matrix; $\bar{f}$ is a residual vector; $\bar{u}$ is a vector of global unknowns.

There is analogue of stiffness matrix in the continual case. We propose that corresponding operator should be called stiffness operator [11, 12, 16]. Direct formulation of this stiffness operator is presented in the distinctive paper.

2. Formulation of a problem for elliptic set of second-order differential equations

Boundary problems for elliptic set of differential equations corresponds to various problems of structural mechanics [8-17]. For instance in elastic theory in arbitrary coordinates. Therefore it is expedient to develop general statements for elliptic set of differential equations. And, thereafter, to directly derive specific formulations for basic problems of elastic theory and heat conduction (or, to be more exact, for their basic elliptic members).

The differential self-conjugated operator for elliptic set of M second-order equations in a space of N-dimensional functions has the form

$$L = \sum_{i=1}^{N}\sum_{j=1}^{N} \partial_i^* A_{i,j} \partial_j u \tag{2.1}$$

where

$$A_{i,j} = \begin{bmatrix} a_{i,j}^{1,1} & a_{i,j}^{1,2} & \dots & a_{i,j}^{1,M} \\ a_{i,j}^{2,1} & a_{i,j}^{2,2} & \dots & a_{i,j}^{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i,j}^{M,1} & a_{i,j}^{M,2} & \dots & a_{i,j}^{M,M} \end{bmatrix} = \{a_{i,j}^{p,q}\}_{p,q=1,\dots,M} \tag{2.2}$$

$$u = [\; u_1 \quad u_2 \quad \dots \quad u_N \;]^T \tag{2.3}$$

We imply here that $A_{i,j} = A_{j,i}^*$; symbol * is conjugation notation.

The formulation (2.1) can be apparently rewritten in matrix form:

$$L = D^* A D \qquad (2.4)$$

where

$$D = \nabla \otimes E_M ; \quad D^* = \nabla^* \otimes E_M \qquad (2.5)$$

We use the following notation:

$$\nabla = \begin{bmatrix} \partial_1 \\ \vdots \\ \partial_N \end{bmatrix} = grad$$

is the gradient;

$$\nabla = -[\, \partial_1 \;\; \dots \;\; \partial_N \,] = -div$$

is the divergence;

E_M is the identity matrix of the M-th order; $\otimes$ is the sign of direct product of matrices i.e.

$$D = \begin{bmatrix} \partial_1 E_M \\ \partial_2 E_M \\ \vdots \\ \partial_N E_M \end{bmatrix} \qquad (2.6)$$

$$D^* = -[\, \partial_1 E_M \;\; \partial_2 E_M \;\; \dots \;\; \partial_N E_M \,] \qquad (2.7)$$

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,M} \\ A_{2,1} & A_{2,2} & \dots & A_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ A_{M,1} & A_{M,2} & \dots & A_{M,M} \end{bmatrix} = \{A_{p,q}\}_{p,q=1,\dots,M} \qquad (2.8)$$

Let us derive basic formulas for operator of boundary problem. Operating on product θu (where u is an arbitrary function; θ is the characteristic function of domain Ω) we get

$$\begin{aligned} L\theta u &= D^* A D \theta u = D^* A[\theta D u + (D^*\theta) u] = \\ &= D^*\theta A D u + D^* A (D^*\theta) u = \qquad (2.9) \\ &= \theta D^* A D u + (D^*\theta) A D u + D^* A (D^*\theta) u, \end{aligned}$$

where $D\theta$ are derivatives of θ;

$$D\theta = \delta_{\tilde{A}} \nu_E \quad D^*\theta = -\delta_{\tilde{A}} \nu_E^* \quad \nu_E = \nu \otimes E_M \qquad (2.10)$$

$\overline{\nu} = [\, \nu_1 \;\; \nu_2 \;\; \dots \;\; \nu_N \,]^T$ is the unit normal direction vector on the domain boundary $\partial\Omega$;

$$\nu_E = \begin{bmatrix} \nu_1 E_M \\ \nu_2 E_M \\ \vdots \\ \nu_N E_M \end{bmatrix} \qquad (2.11)$$

$$\nu_E^* = -[\, \nu_1 E_M \;\; \nu_2 E_M \;\; \dots \;\; \nu_N E_M \,] \qquad (2.12)$$

$\delta_b = \delta(p)$ is the delta function of domain boundary $\partial\Omega$;

$$\delta_b = \partial\theta / \partial\overline{\nu} \qquad (2.13)$$

p(x) = 0 is the equation of domain boundary $\partial\Omega$ [8-17].

Let l is the operator defining natural conditions at the domain boundary $\partial\Omega$

$$l = -\nu_E^* A D = -\sum_{i=1}^{N} \sum_{j=1}^{N} \nu_i A_{i,j} \partial_j \qquad (2.14)$$

Corresponding adjoint operator has the form

$$l = -D^* A \nu_E = \sum_{i=1}^{N} \sum_{j=1}^{N} \partial_j A_{i,j} \nu_i \qquad (2.15)$$

Let us also introduce self-adjoint operator

$$L_0 = D^* \theta A D \qquad (2.16)$$

It is the operator of the second (main) boundary problem.

Using (2.10)-(2.16), we get

$$L_0 u = \theta L u + \delta_\Gamma l u \qquad (2.17)$$

$$L\theta u = L_0 u - l^*(\delta_\Gamma u) \qquad (2.18)$$

$$L\theta u = \theta L u + \delta_\Gamma l u - l^*(\delta_\Gamma u) \qquad (2.19)$$

Formulas (2.17)-(2.19) are basic relations of presenting the operational method. Operators of boundary problems with matched characteristics of boundary conditions can be derived directly from (2.17)-(2.19) [17].

In the distinctive case derivation carries an analogy with derivation of boundary integral equations from the second Green's formula. Namely we partially replace components with functions, which are known from the formulation of boundary problem. The rest of components are transposed from the right-hand to the left-hand side, which can be rewritten in a various forms with the use of (2.17)-(2.18) [17].

Within a domain ($x \in \Omega$) we have the following equation:

$$Lu = F$$

Therefore we can replace component $\theta L u$ with θF.

Let us get operational formulations

$$\tilde{L}u = \tilde{F}$$

for basic boundary conditions with the use of this approach.

The second boundary problem.

$$(l\,u = f,\quad x \in \Gamma = \partial\Omega)$$

Operational formulations have the form

$$L\theta u + l^{*}\delta_{\Gamma}u = \tilde{F} \tag{2.20}$$

or $\quad L_0 u = F \quad$ or $\quad \theta L u + \delta_{\Gamma} l u = \tilde{F} \qquad (2.21)$

i.e. $\quad \tilde{L} = L\theta + l^{*}\delta_{\Gamma} = L_0 = \theta L + \delta_{\Gamma} l \qquad (2.22)$

where

$$\tilde{F} = \theta F + \delta_{\Gamma} f \tag{2.23}$$

The first boundary problem.

$$(u = g,\quad x \in \Gamma = \partial\Omega)$$

Operational formulations have the form

$$L\theta u - \delta_{\Gamma} l u = \tilde{F} \tag{2.24}$$

$$L_0 u - (\delta_{\Gamma} l u + l^{*}(\delta_{\Gamma} u)) = \tilde{F} \tag{2.25}$$

$$\theta L u - l^{*}(\delta_{\Gamma} u) = \tilde{F} \tag{2.26}$$

where

$$\tilde{F} = \theta F - l(\delta_{\Gamma} g) \tag{2.27}$$

Thus

$$\begin{aligned} \tilde{L} &= L\theta - \delta_{\Gamma} l = \\ &= L_0 - \delta_{\Gamma} l - l^{*}\delta_{\Gamma} = \theta L - l^{*}\delta_{\Gamma} \end{aligned} \tag{2.28}$$

Self-conjugacy of the operator of the first boundary problem is apparent from the last formulation. Nevertheless this formulation is inconvenient for numerical applications and corresponding operator of Dirichlet's problem is not positive define [17].

The third boundary problem.

$$(lu + h(u - g) = b,\quad x \in \Gamma = \partial\Omega)$$

Using boundary conditions we can write

$$\delta_{\Gamma} l u = -\delta_{\Gamma} h u + \delta_{\Gamma}(b + hg) \tag{2.29}$$

Operational formulations have the form

$$L\theta u + l^{*}\delta_{\Gamma} u + \delta_{\Gamma} h u = \tilde{F} \tag{2.30}$$

$$L_0 u + \delta_{\Gamma} h u = \tilde{F} \tag{2.31}$$

$$\theta L u + \delta_{\Gamma} l u + \delta_{\Gamma} h u = \tilde{F} \tag{2.32}$$

where

$$F = \theta F + \delta_{\Gamma}(b + hg) \tag{2.33}$$

$$\begin{aligned} \tilde{L} &= L\theta + l^{*}\delta_{\Gamma} + \delta_{\Gamma} h = \\ &= L_0 + \delta_{\Gamma} h = \theta L - \delta_{\Gamma} l + \delta_{\Gamma} h \end{aligned} \tag{2.34}$$

We must just note that operator of the considering problem is the operator if the second boundary problem with diagonal component.

The third boundary problem is vital for practice due to direct description of physical processes at the domain boundary (Winkler foundation in the problem of elastic theory or heat exchange in the problem of heat conduction, for instance) and simulation of basic boundary conditions. We have natural boundary conditions if $h = 0$ and main boundary conditions if $h = 1$. In case of $h = 1$ fixing conditions are simulated by supports of great stiffness. This approach corresponds to so-called penalty method and it is especially convenient for numerical and computer realization. Thus, the third boundary problem is the most universal and preferable for applications [11, 12].

The mixed boundary problem.

$$\begin{cases} lu = f, & x \in \Gamma_2 \\ u = g, & x \in \Gamma_1 \end{cases},\qquad \Gamma_1 \cup \Gamma_2 = \Gamma = \partial\Omega$$

In the case of mixed boundary problem we have various boundary conditions at different parts of the domain boundary and finally obtain combination of equations, which have already been formulated above.

Let χ be characteristic function of the part of domain boundary with natural boundary conditions and let $\overline{\chi}$ be its corresponding complement,

$$\chi\Gamma = \Gamma_2;\quad \overline{\chi}\Gamma = \Gamma_1;\quad \chi + \overline{\chi} = 1 \tag{2.35}$$

Using (2.19) we get

$$\begin{aligned} L\theta\theta &= \theta L u + \chi\delta_{\Gamma} f + \overline{\chi}\delta_{\Gamma} l u - \\ &- l^{*}\overline{\chi}\delta_{\Gamma} g - l^{*}\chi\delta_{\Gamma} u. \end{aligned} \tag{2.36}$$

Therefore we have

$$L\theta\theta - \overline{\chi}\delta_{\Gamma} l u + l^{*}\chi\delta_{\Gamma} u = \tilde{F} \tag{2.37}$$

$$L_0 u - \overline{\chi}\delta_{\Gamma} l u - l^{*}\overline{\chi}\delta_{\Gamma} u = \tilde{F} \tag{2.38}$$

$$\theta L u + \overline{\chi}\delta_{\Gamma} l u - l^{*}\overline{\chi}\delta_{\Gamma} u = \tilde{F} \tag{2.39}$$

$$\begin{aligned} L &= L\theta - \overline{\chi}\delta_{\Gamma} l + l^{*}\chi\delta_{\Gamma} = \\ &= \theta L + \overline{\chi}\delta_{\Gamma} l - l^{*}\overline{\chi}\delta_{\Gamma} = \mathsf{L}_0 - \overline{\chi}\delta_{\Gamma} l - l^{*}\overline{\chi}\delta_{\Gamma} \end{aligned} \tag{2.40}$$

Each of presenting operational formulations of boundary problems is the united expression, including condition within the domain and boundary conditions. These formulations also provide matched weight factors and they can be considered at arbitrary embordering domain. It is especially significant for numerical realization. In this connection we recommend application of so-called method of extended domain. In terms of the distinctive method all problems are considered at extended domain of arbitrary shape, particularly elementary (for instance, parallel-piped, cylinder and others). This leads to convenient mathematical formulas, effective computational schemes and algorithms, simple data processing and so on [8-17].

3. Symmetric formulation of a mixed boundary problem

In accordance with given boundary condition we get

$$\theta Lu = \theta F;\quad \delta_\Gamma \chi l u = \delta_\Gamma \chi f;\quad -l^* \delta_\Gamma \overline{\chi} u = l^* \delta_\Gamma \overline{\chi} g \tag{3.1}$$

If we combine equations (3.1) we obtain

$$\theta Lu + \delta_\Gamma \chi l u - l^* \delta_\Gamma \overline{\chi} u = \theta F + \delta_\Gamma \chi f + l^* \delta_\Gamma \overline{\chi} g \tag{3.2}$$

Using (2.17)-(2.19) we have

$$\theta Lu = L_0 u - \delta_\Gamma l u$$

and consequently

$$L_0 u - \delta_\Gamma l u + \delta_\Gamma \chi l u - l^* \delta_\Gamma \overline{\chi} u = \widetilde{F} \tag{3.3}$$

Taking into account that

$$-\delta_\Gamma l u + \delta_\Gamma \chi l u = -\delta_\Gamma \overline{\chi} l u$$

we get

$$L_0 u - \delta_\Gamma \overline{\chi} l u - l^* \delta_\Gamma \overline{\chi} u = \widetilde{F} \tag{3.4}$$

where

$$\widetilde{F} = \theta F + \delta_\Gamma \chi f + l^* \delta_\Gamma \overline{\chi} g \tag{3.5}$$

Equation (3.4) can be rewritten in the following form

$$L_0 u - (\widetilde{l}_\Gamma + \widetilde{l}_\Gamma^*) u = \widetilde{F} \tag{3.6}$$

or

$$L_0 u - l_\Gamma u = \widetilde{F} \tag{3.7}$$

where

$$\widetilde{l}_\Gamma = \delta_\Gamma \overline{\chi} l \tag{3.8}$$

$$l_\Gamma = \widetilde{l}_\Gamma + \widetilde{l}_\Gamma^* \tag{3.9}$$

We propose to call operator $L_0 - l_\Gamma$ as stiffness operator [11, 12] on the analogy of technics of finite element method [2, 6].

The presenting operational formulation (3.7) of mixed boundary problem is symmetric.

4. Variational formulation of a mixed boundary problem

On the basis of (3.6) we have

$$\Phi(u) = \frac{1}{2}\int_\Omega (L_0 u, u)dx - \int_\Omega (F, u)dx - \int_{\Gamma_1} (\overline{\chi} l u, \overline{\chi} u)dx + \int_{\Gamma_2} (f, \chi u)dx - \int_{\Gamma_1} (g, \overline{\chi} l u)dx. \tag{4.1}$$

5. Several examples of formulations

Three-dimensional problem of elasticity.

We have [17]

$$L_0 = \sum_{j=1}^{3} \partial_j^* \overline{\mu} \partial_j \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \partial_1^* \overline{\mu} \partial_1 & \partial_2^* \overline{\mu} \partial_1 & \partial_3^* \overline{\mu} \partial_1 \\ \partial_1^* \overline{\mu} \partial_2 & \partial_2^* \overline{\mu} \partial_2 & \partial_3^* \overline{\mu} \partial_2 \\ \partial_1^* \overline{\mu} \partial_3 & \partial_2^* \overline{\mu} \partial_3 & \partial_3^* \overline{\mu} \partial_3 \end{bmatrix} + \begin{bmatrix} \partial_1^* \overline{\lambda} \partial_1 & \partial_1^* \overline{\lambda} \partial_2 & \partial_1^* \overline{\lambda} \partial_3 \\ \partial_2^* \overline{\lambda} \partial_1 & \partial_2^* \overline{\lambda} \partial_2 & \partial_2^* \overline{\lambda} \partial_3 \\ \partial_3^* \overline{\lambda} \partial_1 & \partial_3^* \overline{\lambda} \partial_2 & \partial_3^* \overline{\lambda} \partial_3 \end{bmatrix} \tag{5.1}$$

Here λ, μ are Lame coefficients; $\overline{\lambda}$, $\overline{\mu}$ are Lame coefficients, defined at extended domain $\omega \supset \Omega$ ($\overline{\lambda} = \overline{\mu} = 0$ outside Ω),

$$\overline{\lambda} = \theta\lambda;\quad \overline{\mu} = \theta\mu;\quad \partial_j = \partial/\partial x_j \tag{5.2}$$

Operators of the problem have form

$$Lu = \sum_{j=1}^{N} \partial_j \sigma_{i,j} = -F_i,\quad x \in \Omega \tag{5.3}$$

$$lu = \sum_{j=1}^{N} \nu_j \sigma_{i,j} = -f_i,\quad x \in \Gamma_2 \tag{5.4}$$

Formulas for strain and stress components have the form:

$$\sigma_{i,j} = \delta_{i,j}\lambda\varepsilon + 2\mu\varepsilon_{i,j} \quad (5.5)$$

$$\varepsilon_{i,j} = \frac{1}{2}(\partial_i u_j + \partial_j u); \quad \varepsilon = \sum_{i=1}^{N} \varepsilon_{i,i} \quad (5.6)$$

Where u_i–displacement components; $\delta_{i,j}$ is Chronicler's symbol.

One-dimensional problem of compression of column. We have [15]

$$d^2\theta u = F,\ x \in (0,a);\ \chi l u = \chi f,\ x \in \Gamma_2;$$
$$\bar{\chi} u = \bar{\chi} g,\ x \in \Gamma_1 \quad (5.7)$$

where

$$d = d/dx; \quad d^2 = d^2/dx^2$$

Values of inside normal at points $x = 0$ and $x = a$ are $\nu = 1$ and $\nu = -1$ consequently.
Operators of the problem have form

$$l = \nu d; \quad L_0 = d\theta d \quad (5.8)$$

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