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ABOUT CORRECT METHOD OF ANALYTICAL SOLUTION OF MULTIPOINT BOUNDARY PROBLEMS OF STRUCTURAL MECHANICS FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT COEFFICIENTS

Abstract

This paper is devoted to correct method of analytical solution of multipoint boundary problems of structural mechanics for systems of ordinary differential equations with piecewise constant coefficients. Its major peculiarities include uni-versality, computer-oriented algorithm involving theory of distributions, computational stability, optimal conditionality of resultant systems and partial Jordan decomposition of matrix of coefficients, eliminating necessity of calculation of root vectors.

Keywords: correct analytical solution, multipoint boundary problem, discrete-continual methods, structural mechanics, system of ordinary differential equations, piecewise constant coefficients

1. Formulation of the Problem

Piecewise invariability of physical and geometrical parameters in one dimension exists in various problems of analysis of structures and their mathematical models. We should mention here in particular such vital objects as beams, strip foundations, thin-walled bars, deep beams, plates, shells, high-rise buildings, extensional buildings, pipelines, rails, dams and others [1, 2, 3]. Analytical solution is apparently preferable in all aspects for qualitative analysis of calculation data. It allows investigator to consider boundary effects when some components of solution are rapidly varying functions. Due to the abrupt decrease inside of mesh elements in many cases their rate of change can't be adequately considered by conventional numerical methods while analytics enables study. Another feature of the proposing method is the absence of limitations on lengths of structures. Hence it appears that in this context so-called discrete-continual methods of structural analysis (especially discrete-continual finite element method) are peculiarly relevant [1, 2, 3]. Generally, discrete-continual formulations are contemporary mathematical models which currently becoming available for computer realization.

Discrete-continual methods are reduced at some stage to the solution of multipoint boundary problems of structural mechanics for systems of ordinary differential equations with piecewise constant coefficients. Conventional formulation of multipoint boundary problem of this type has the form

$$\bar{y}^{(1)} - A_k \bar{y} = \bar{f}_k, \quad x \in (x_k^b, x_{k+1}^b), \quad k = 1, 2, \ldots, n_k - 1; \qquad (1.1)$$

$$B_k^- \bar{y}(x_k^b - 0) + B_k^+ \bar{y}(x_k^b + 0) = \bar{g}_k^- + \bar{g}_k^+, \quad k=2, \ldots, n_k - 1; \qquad (1.2)$$

$$B_1^+ \bar{y}(x_1^b + 0) + B_{n_k}^- \bar{y}(x_{n_k}^b - 0) = \bar{g}_1^+ + \bar{g}_{n_k}^-, \qquad (1.3)$$

where

$\bar{y} = \bar{y}(x) = [y_1(x) \;\; y_2(x) \;\; \ldots \;\; y_n(x)]^T$

is the desirable vector function;

$x_k^b, \quad k=1, \ldots, n_k$

are coordinates of boundary points;

A_k, $k = 1, 2, \ldots n_k - 1$

are matrices of constant coefficients of order n;

$\bar{f}_k = \bar{f}_k(x) = [\, f_{k,1}(x) \;\; f_{k,2}(x) \;\ldots f_{k,n}(x)\,]^T$

$k = 1, 2, \ldots n_k - 1$

are right-side vector functions

$B_k^-,\; B_k^+,\;\; k = 2, \ldots, n_k - 1$ and $B_1^+,\; B_{n_k}^-$

are matrices of boundary conditions of order n at point

$x_k^b;\;\; \bar{g}_k^-,\; \bar{g}_k^+,\;\; k = 2, \ldots, n_k - 1$ and $\bar{g}_1^+,\; \bar{g}_{n_k}^-$

are right-side vectors of boundary conditions at point

$x_k^b;\;\; \bar{y}^{(1)} = \bar{y}^{(1)}(x) = d\bar{y} / dx$.

Solution of multipoint boundary problem of this type in structural mechanics is accentuated by numerous factors. They include boundary effects (stiff systems) and considerable number of differential equations (several thousands). Moreover, matrices of coefficients of a system normally have eigenvalues of opposite signs and corresponding Jordan matrices are not diagonal. Special method of solution of multipoint boundary problems for systems of ordinary differential equations with piecewise constant coefficients in structural mechanics has been developed. Not only does it overcome all difficulties mentioned above but its major peculiarities also include universality, computer-oriented algorithm, computational stability, optimal conditionality of resultant systems and partial Jordan decomposition of matrix of coefficients, eliminating necessity of calculation of root vectors.

2. Jordan Decomposition of Matrices of Coefficients

Jordan decomposition of matrix A_k has the form

$$A_k = T_k J_k T_k^{-1}, \tag{2.1}$$

where

$$J_k = \{J_{k,1},\, J_{k,2},\, \ldots,\, J_{k,u_k}\}; \tag{2.2}$$

T_k is the matrix of order n, which columns are eigenvectors and root vectors of matrix A_k; J_k is Jordan matrix of order n; $J_{k,p}$ is Jordan cell corresponding to eigenvalue $\lambda_{k,p}$; $\dim J_{k,p} = m_{k,p}$.

As we have already mentioned above, specificity of problems of structural mechanics comprises in presence of multiple eigenvalues of matrix A_k and consequently in necessity of calculation of root vectors. However at the present time there are no effective numerical method of calculation of Jordan decomposition in the general case [4]. Meanwhile the number of multiple eigenvalues in the considering type of problems is normally limited. Besides these multiple eigenvalues are generally zeros. In this connection special alternative approach to solution has been developed.

3. Partial Jordan Decomposition

Partial Jordan decomposition is based on computation of right and left eigenvectors of matrix A_k.

$$A_k = A_{k,1} + A_{k,2}; \tag{3.1}$$

$$A_{k,1} = T_{k,1} J_{k,1} \tilde{T}_{k,1};\;\; A_{k,2} = A_k - A_{k,2}; \tag{3.2}$$

$T_{k,1}$ is the matrix containing right eigenvectors corresponding to non-zero eigenvalues of matrix A_k; $\tilde{T}_{k,1}$ is the matrix containing left eigenvectors corresponding to non-zero eigenvalues of matrix A_k; $J_{k,1}$ is diagonal Jordan matrix corresponding to non-zero eigenvalues of matrix A_k; $A_{k,2}$ is the part of matrix A_k corresponding to prime and multiple zero eigenvalues. It is necessary to note here that matrices $T_{k,1}$ and $\tilde{T}_{k,1}$ in general case are rectangular.

4. Construction of Projectors

Eigenvalues $\lambda_{k,p}$, $p = 1, \ldots, u_k$ are renumbered according to the condition

$$\left.\begin{aligned} &\forall \lambda_{k,p},\quad p = 1, \ldots, l_k \quad \exists m_{k,p} = 1 \\ &\forall \lambda_{k,p},\quad p = l_k + 1, \ldots, u_k \quad \exists m_{k,p} > 1 \end{aligned}\right\}, \tag{4.1}$$

where

$$l_k = \sum_{p=1}^{u_k} \delta_{1,m_{k,p}}, \tag{4.2}$$

where $\delta_{i,j}$ is Kronecker delta.

Due to distinctive procedure, we should properly modify matrices $T_{k,1}$, $\tilde{T}_{k,1}$ and $J_{k,1}$.

Let $P_{k,1}$ and $P_{k,2}$ be projectors to subspaces of left and right eigenvectors and root vectors of matrix A_k corresponding to non-zero and zero eigenvalues. They may be denoted as

$$P_{k,1} = T_{k,1}(\tilde{T}_{k,1} T_{k,1})^{-1} \tilde{T}_{k,1};\;\; P_{k,2} = E - P_{k,1}, \tag{4.3}$$

where E is identity matrix.

5. Construction of Fundamental Matrix-function of System of Equations

After sorting and biorthogonalization of eigenvectors and eigenvalues fundamental matrix-function $\varepsilon_k(x)$ of system from Eq. (1.1) for arbitrary k is constructed in the special form convenient for problems of structural mechanics

$$\begin{aligned}\varepsilon_k(x) &= T_{k,1}\tilde{\varepsilon}_{k,0}(x)\tilde{T}_{k,1} + \\ &+\chi(x,0)[P_{k,2} + \sum_{j=1}^{m_{k,\max}-1}\frac{x^j}{j!}A_{k,2}^j],\end{aligned} \tag{5.1}$$

where

$$m_{k,\max} = \max_{l\le i\le u_k} m_{k,i}; \tag{5.2}$$

$$\chi(x,\lambda_{k,p}) = \begin{cases}\mathrm{sign}(x)\theta(-\mathrm{Re}(\lambda_{k,p})x), & \lambda_{k,p}\neq 0\\ 0.5\cdot\mathrm{sign}(x), & \lambda_{k,p}=0;\end{cases} \tag{5.3}$$

$$\mathrm{sign}(x) = \begin{cases}1, & x>0\\ -1, & x<0;\end{cases} \tag{5.4}$$

$$\theta(x) = \begin{cases}1, & x>0\\ 0, & x<0;\end{cases} \tag{5.5}$$

$$\begin{aligned}\tilde{\varepsilon}_{k,0}(x) = diag\{&\chi(x,\lambda_{k,1})\exp(\lambda_{k,1}x),\ \ldots,\\ &\chi(x,\lambda_{k,l_k})\exp(\lambda_{k,l_k}x)\}.\end{aligned} \tag{5.6}$$

It should be stated that the sum in the right side of Eq. (5.1) contains four or lower components and corresponds to so-called "beam" part of solution of system.

6. General Solution of the Problem

Solution of considering problem (Eq. (1.1), Eq. (1.2), Eq. (1.3)) on the interval (x_k^b, x_{k+1}^b) is defined by formula

$$\begin{aligned}\bar{y}_k(x) &= (\varepsilon_k(x-x_k^b) - \varepsilon_k(x-x_{k+1}^b))\bar{C}_k + \\ &+\varepsilon_k * \bar{f}_k, \quad x\in(x_k^b, x_{k+1}^b),\end{aligned} \tag{6.1}$$

where $\bar{C}_k$ is the vector of constants of order n; $*$ is convolution notation;

$$\bar{f}_k(x) \equiv f(x)\theta(x, x_k^b, x_{k+1}^b); \tag{6.2}$$

$$\theta(x, x_k^b, x_{k+1}^b) = \begin{cases}1, & x\in(x_k^b, x_{k+1}^b)\\ 0, & x\notin(x_k^b, x_{k+1}^b).\end{cases} \tag{6.3}$$

We can rewrite Eq. (6.1) in the form

$$\bar{y}_k(x) = E_k(x)\bar{C}_k + \bar{S}_k,\ x\in(x_k^b, x_{k+1}^b); \tag{6.4}$$

$$E_k(x) = \varepsilon_k(x-x_k^b) - \varepsilon_k(x-x_{k+1}^b); \tag{6.5}$$

$$\bar{S}_k(x) = \varepsilon_k * \bar{f}_k. \tag{6.6}$$

Substituting Eq. (6.4) in Eq. (1.2) and Eq. (1.3) and taking into account that

$$\bar{y}(x_k-0) = \bar{y}_{k-1}(x_k-0),\ k=2,\ \ldots,\ n_k; \tag{6.7}$$

$$\bar{y}(x_k+0) = \bar{y}_k(x_k+0), \quad k=1,\ \ldots,\ n_k-1; \tag{6.8}$$

$$\begin{aligned}E_{k-1}(x_k^b-0) = \varepsilon_{k-1}(h_{k-1}^b) - \varepsilon_{k-1}(-0),\\ k=2,\ \ldots,\ n_k;\end{aligned} \tag{6.9}$$

$$\begin{aligned}E_k(x_k^b+0) = \varepsilon_k(+0) - \varepsilon_k(-h_k^b),\\ k=1,\ \ldots,\ n_k-1;\end{aligned} \tag{6.10}$$

$$h_k^b = x_{k+1}^b - x_k^b, \quad k=1,\ \ldots,\ n_k-1 \tag{6.11}$$

we have the following system of linear algebraic equations for $\bar{C}_k$, $k=1,\ldots,n_k-1$

$$K\bar{C} = \bar{G}, \tag{6.12}$$

where matrix K can be divided into so-called main K^0 and additional K^1 members,

$$K^0 = \begin{bmatrix} K_{1,1}^0 & 0 & 0 & \ldots & 0 & K_{1,n_k-1}^0 \\ K_{2,1}^0 & K_{2,2}^0 & 0 & \ldots & 0 & 0 \\ 0 & K_{3,2}^0 & K_{3,3}^0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & K_{n_k-1,n_k-2}^0 & K_{n_k-1,n_k-1}^0 \end{bmatrix} \tag{6.13}$$

$$K^1 = \begin{bmatrix} K_{1,1}^1 & 0 & 0 & \ldots & 0 & K_{1,n_k-1}^1 \\ K_{2,1}^1 & K_{2,2}^1 & 0 & \ldots & 0 & 0 \\ 0 & K_{3,2}^1 & K_{3,3}^1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & K_{n_k-1,n_k-2}^1 & K_{n_k-1,n_k-1}^1 \end{bmatrix} \tag{6.14}$$

$$K^{0}_{k,k\text{-}1} = -B^{-}_{k}\ \varepsilon_{k-1}(-0)\,; \qquad (6.15)$$

$$K^{0}_{k,k} = B^{+}_{k}\varepsilon_{k}(+0)\,; \qquad (6.16)$$

$$K^{0}_{1,1} = B^{+}_{1}\ \varepsilon_{1}(+0)\,; \qquad (6.17)$$

$$K^{0}_{1,n_k-1} = -B^{-}_{n_k}\varepsilon_{n_k-1}(-0)\,; \qquad (6.18)$$

$$K^{1}_{k,k\text{-}1} = B^{-}_{k}\ \varepsilon_{k-1}(h^{b}_{k-1})\,; \qquad (6.19)$$

$$K^{1}_{k,k} = -B^{+}_{k}\varepsilon_{k}(-h^{b}_{k})\,; \qquad (6.20)$$

$$K^{1}_{1,1} = -B^{+}_{1}\ \varepsilon_{1}(-h^{b}_{1})\,; \qquad (6.21)$$

$$K^{1}_{1,n_k-1} = B^{-}_{n_k}\varepsilon_{n_k-1}(h^{b}_{n_k-1})\,; \qquad (6.22)$$

$$\overline{G} = [\ \overline{G}^{T}_{1}\quad \overline{G}^{T}_{2}\quad \ldots\quad \overline{G}^{T}_{n_k-1}\ ]^{T}\,; \qquad (6.23)$$

$$\overline{C} = [\ \overline{C}^{T}_{1}\quad \overline{C}^{T}_{2}\quad \ldots\quad \overline{C}^{T}_{n_k-1}\ ]^{T}\,; \qquad (6.24)$$

$$\begin{aligned}\overline{G}_{1} = \overline{g}^{+}_{1} + \overline{g}^{-}_{n_k} - B^{+}_{1}\overline{S}_{1}(x^{b}_{1}+0) - \\ - B^{-}_{n_k}\overline{S}_{n_k-1}(x^{b}_{n_k}-0);\end{aligned} \qquad (6.25)$$

$$\begin{aligned}\overline{G}_{k} = \overline{g}^{-}_{k} + \overline{g}^{+}_{k} - \\ -B^{-}_{k}\overline{S}_{k-1}(x^{b}_{k}-0) - B^{+}_{k}\overline{S}_{k}(x^{b}_{k}+0), \\ k = 2,\ \ldots,\ n_k - 1.\end{aligned} \qquad (6.26)$$

Symbol $\otimes$ imply direct product of matrices. It is necessary to note that matrices $\varepsilon_k(+0)$ and $\varepsilon_k(-0)$ are independent of x.

We find it vital to note that diagonal blocks of matrix K are practically singular. This fact leads to several problems. Iterative methods of solution can't be applied in particular. Gaussian elimination method with pivoting is required. It is useful to specify ways of disposal of this disadvantage.

Let us transform considering system of equation as follows: each equation of system, since the first (and finishing next to last), we will replace with the sum of this equation with the subsequent (instead of the initial first equation we take the sum initial the first with initial the second, instead of initial the second – the sum initial the second with initial the third and so on). Instead of initial last equation we take the sum of the initial last with initial the first. Finally we have:

$$K = \\ = \begin{bmatrix} \tilde{K}_{1,1} & \tilde{K}_{1,2} & 0 & 0 & \ldots & 0 & 0 & \tilde{K}_{1,n_k-1} \\ \tilde{K}_{2,1} & \tilde{K}_{2,2} & \tilde{K}_{2,3} & 0 & \ldots & 0 & 0 & 0 \\ 0 & \tilde{K}_{3,2} & \tilde{K}_{3,3} & \tilde{K}_{3,4} & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \tilde{K}_{n_k-1,1} & 0 & 0 & 0 & \ldots & 0 & \tilde{K}_{n_k-1,n_k-2} & \tilde{K}_{n_k-1,n_k-1} \end{bmatrix}. \qquad (6.27)$$

Thus we removed the singularity mentioned above.

Acknowledgments

This work was financially supported by the Grant of the President of the Russian Federation for Leading Scientific Schools (SS-8684.2010.8), Grant of Russian Academy of Architecture and Construction Sciences (2.3.9), Analytical Departmental Target Program "Development of Scientific Potential of Higher Education" (Project 2.1.2/12148).

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